MATH 8

HOMEWORK 7 PARTIAL SOLUTIONS

1. Find the prime factorization of 111111.
   Solution: $111111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$
2. (a) Which positive integers have exactly three positive divisors?
   Solution: $n = p^2$, where p is prime.
   (b) Which positive integers have exactly four positive divisors?
   Solution: $n = p_1 p_2$, where p_1 and p_2 are distinct primes, and $n = q^3$, where q is prime.
   (c) Suppose $n \geq 2$ is an integer with the property that whenever a prime p divides n, p^2 also divides n (i.e. all primes in the prime factorization of n appear at least to the power 2). Prove that n can be written as the product of a square and a cube.

   Proof. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of n, where each p_i is a distinct prime and $a_i \geq 2$ for all i. It suffices to prove that we can find a factorization of n in which the exponent of each factor is either a multiple of 2 or a multiple of 3. So, if every exponent a_i is already either a multiple of 2 or a multiple of 3, then we are happy and done! Therefore, we suppose there is some exponent a_k that is neither a multiple of 2 nor a multiple of 3 (5 is an example of such a positive integer). Note that a_k is an odd integer greater than 3. Hence $a_k - 3$ is even. Thus, if there is any prime power $p_k^{a_k}$ in the factorization above, where a_k is neither a multiple of 2 nor 3, we write $p_k^{a_k} = p_k^{a_k - 3} p_k^3$. Therefore, the prime factorization of n can be written in such a way that each exponent is either a multiple of 2 or a multiple of 3 (and note that now this factorization may not have each prime distinct). □

4. Prove that $lcm(a, b) = \dfrac{ab}{gcd(a, b)}$ for any positive integers a, b without using prime factorization.

Proof. This is a sketch of the proof. You are left to fill in the details.

Let's start with basic notation. Let $m = lcm(a, b)$ and $d = gcd(a, b)$. We want to show that $ab = dm$.

(a) First show that since d divides a and d divides b, then d must also divide the product ab.
(b) Once you've shown the above, this means (by definition) that we can write $ab = dn$ for some integer n. Now the goal of the problem is to show that n must actually be equal to m.
(c) Next, show that n is a common multiple of a and b. That is, show a divides n and b divides n.
(d) Finally, show that n divides m.

(e) Note that the previous two steps yield $n = m$. From item (c), we an conclude that $m \leq n$ (since m is the LEAST common multiple of a and b it must be less than or equal to every common multiple of a and b). From item (d) we can conclude that $n \leq m$. Thus, these two inequalities yield $n = m$.

□

6. On your own or discuss in section.
8. Find all solutions $x, y \in \mathbb{Z}$ to the following Diophantine equations:
   (a) $x^2 = y^3$
   Solution: Any integer that is both a square and a cube is a $6th$ power, and conversely, every integer that is a $6th$ power is both a square and a cube. So the solutions are $x = a^3$ and $y = a^2$ for every integer a.
   (b) $x^2 - x = y^3$
   Solution: Factor the left hand side as $x(x-1)$. The two integers x and $x-1$ are coprime, and their product is a cube. Thus, by Proposition 12.4, both x and $x-1$ are cubes, and in particular, their difference is 1. The only integers x that make this true are $x = 0, 1$. Hence the solutions are $x = 0, y = 0$ and $x = 1, y = 0$.
   (c) $x^2 = y^4 - 77$
   Solution: $x = 4, y = 3$ is one solution. Are there any others?
   (d) $x^3 = 4y^2 + 4y - 3$
   Solution: Factor the right hand side to obtain $x^3 = (2y-1)(2y+3)$ now mimic Example 12.1.